

# Quantum non-demolition measurement saturates fidelity trade-off

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A general quantum measurement on an unknown quantum state enables us to estimate what the state originally was. Simultaneously, the measurement has a destructive effect on a measured quantum state which is reflected by the decrease of the output fidelity. We show for any  $d$ -level system that quantum non-demolition (QND) measurement controlled by a suitably prepared ancilla is a measurement in which the decrease of the output fidelity is minimal. The ratio between the estimation fidelity and the output fidelity can be continuously controlled by the preparation of the ancilla. Different measurement strategies on the ancilla are also discussed. Finally, we propose a feasible scheme of such a measurement for atomic and optical 2-level systems based on basic controlled-NOT gate.

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Measurement in quantum mechanics changes drastically measured quantum state. Moreover, this change cannot be done arbitrarily small. This main feature of quantum measurement can be simply proved by performing the estimation of the state after the measurement. At first sight this is a negative effect which does not allow many operations well known from classical physics. Fortunately, there is also a positive aspect of this property. In principle, it can be exploited to make communication between two distant stations secure against eavesdropping attacks. Namely, secret information can be sent by quantum states in such a way that any measurement on the transmitted states can be detected and consequently any attack on the link can be revealed [1]. This property represents a fundamental distinction between quantum measurement and classical measurement that can be made in principle state non-destructive. Such an ideal classical measurement has a quantum analogue called *quantum non-demolition* (QND) measurement [2]. The QND measurement is non-destructive in the sense that it preserves probabilistic distribution of so called *non-demolition* variable of the measured system and simultaneously, the measurement results give a perfect copy of the non-demolition variable statistics. From this point of view, they can be used as a perfect distributor of information encoded in the non-demolition variable of a quantum state. All noise arising in the measurement process is transferred to the complementary variables. The present work is devoted to (1) the analysis of the fundamental property of the QND measurement and (2) to the feasible application of the QND measurement for optimal distribution of information encoded in an unknown system variable.

Suppose Alice is given a  $d$ -level quantum system  $S$  (qudit  $S$ ) in an *unknown* pure state  $|\psi\rangle_S$  and she sends the state to Bob. Suppose there is Eve between Alice and Bob that wants to guess this state whereas disturbing it to the least possible extent. For this purpose Eve can measure the state directly by a projective measurement

and based on the outcomes of the measurement she can guess the state. Alternatively, Eve can guess the state from measurement on an ancillary system that previously interacted with the original state. Both the strategies produce two states, an estimate of the original state that is held by Eve  $\rho_{\text{est}}$  and an output state  $\rho_{\text{out}}$  after the measurement that continues toward Bob. The quality of Eve's guesses can be characterized by the mean fidelity  $G$  (estimation fidelity) defined as  $G = \int \langle \psi | \rho_{\text{est}} | \psi \rangle d\psi$  where  $\int d\psi$  is the integral over the space of pure states and  $d\psi$  is the measure invariant with respect to unitary transformations. The perturbation introduced by Eve to the original state can be characterized by the mean fidelity  $F$  (output fidelity) of the output state  $F = \int \langle \psi | \rho_{\text{out}} | \psi \rangle d\psi$ . According to the laws of quantum mechanics the fidelities  $F$  and  $G$  must satisfy the following inequality [3]:

$$\sqrt{F - \frac{1}{d+1}} \leq \sqrt{G - \frac{1}{d+1}} + \sqrt{(d-1) \left( \frac{2}{d+1} - G \right)}. \quad (1)$$

The inequality sets a tightest bound between the mean fidelity  $G$  of estimation of an unknown state from a general deterministic quantum operation on a single qudit and the mean fidelity  $F$  of the state after the operation. Particularly important are quantum operations that saturate the inequality (1). Namely, these operations introduce the least possible disturbance to the original state in the sense that for a given estimation fidelity  $G$  they provide the highest possible output fidelity  $F$ .

In this article we show generally for a qudit that a perfect QND measurement randomly performed along all the basis in the Hilbert space which is controlled by a quantum state of ancilla saturates the inequality (1). In particular, such the QND measurement for a single qubit can be implemented by the basic controlled-NOT (CNOT) operation. The perfect QND measurement means that Eve has a perfect copy of statistics of the non-demolition variable. Further, we discuss in detail an imperfect QND

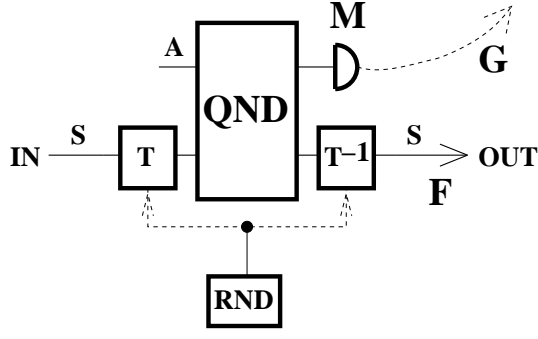


FIG. 1: The scheme of optimal measurement with a minimal disturbance: QND – QND interaction, T – twirling operation, RND – random-number generator, M – state discriminator.

measurement and we compare influence of different discriminations of ancillary states. Finally, we shortly summarize feasible experimental implementations of the proposed measurement scheme and discuss open problems stimulating future research.

Our protocol depicted in Fig. 1 consists of two steps. At the outset we consider the protocol without twirling operations  $T$  and  $T^{-1}$ . In the first step the qudit  $S$  in an unknown state  $|\psi\rangle_S$  is coupled by the two-qudit unitary interaction  $U$  to another ancillary qudit  $A$ . In the second step the information about the state  $|\psi\rangle_S$  is gained from a suitable projective measurement on the ancilla and it is converted into the state of another qudit  $E$ . In order the operation to saturate the inequality (1) the interaction  $U$  must satisfy two following conditions. First, there must exist a normalized vector  $|\mu\rangle_A$  of the ancilla  $A$  such that

$$|\psi\rangle_S |\mu\rangle_A \xrightarrow{U} \sum_{i=1}^d c_i |a_i\rangle_S |\mu_i\rangle_A, \quad (2)$$

where  $\{|a_i\rangle_S\}_{i=1}^d$  and  $\{|\mu_i\rangle_A\}_{i=1}^d$  are the orthonormal bases in the state spaces of the qudit  $S$  and  $A$ , respectively, and  $c_i = \langle a_i | \psi \rangle_S$ . The interaction (2) represents a perfect QND coupling. Second, there must exist a normalized vector  $|\kappa\rangle_A$  that “switches off” the interaction  $U$ , i. e.

$$|\psi\rangle_S |\kappa\rangle_A \xrightarrow{U} |\psi\rangle_S \frac{1}{\sqrt{d}} \sum_{i=1}^d |\mu_i\rangle_A. \quad (3)$$

Obviously, performing the measurement on the ancilla after the transformation (3) in the basis  $\{|\mu_i\rangle_A\}_{i=1}^d$  gives no information on the input state. Now we show that if an unitary interaction  $U$  satisfies the two conditions then it can be used to construct a quantum operation that saturates the inequality (1). Moreover, the flow of information between the qudit  $E$  and the original qudit  $S$  can be controlled by the preparation of the ancilla  $A$ . Let us assume that the ancilla is prepared in the superposition

$$|\tau\rangle_A = \alpha |\mu\rangle_A + \beta |\kappa\rangle_A, \quad (4)$$

where  $\alpha$  and  $\beta$  are positive real numbers satisfying the normalization condition  $\alpha^2 + \beta^2 + 2\alpha\beta/\sqrt{d} = 1$ . The interaction  $U$  transforms the state as

$$|\psi\rangle_S |\tau\rangle_A \xrightarrow{U} \sum_{i=1}^d c_i |a_i\rangle_S \left[ \alpha |\mu_i\rangle_A + \frac{\beta}{\sqrt{d}} \sum_{j=1}^d |\mu_j\rangle_A \right]. \quad (5)$$

Then, the ancilla is measured in the basis  $\{|\mu_i\rangle_A\}_{i=1}^d$ . If the ancilla is found in the state  $|\mu_r\rangle_A$  then Eve prepares her qudit  $E$  in the state  $|a_r\rangle_E$ . The proposed scheme realizes a general quantum operation on qudit  $S$  that can be described by a set of operators  $A_r$ , ( $r = 1, \dots, d$ ). In the basis  $\{|a_i\rangle_S\}_{i=1}^d$  the operators are represented by diagonal matrices with elements

$$(A_r)_{ij} = \left( \alpha \delta_{ir} + \frac{\beta}{\sqrt{d}} \right) \delta_{ij}, \quad (6)$$

where  $i, j = 1, \dots, d$  and  $\delta_{ij}$  is the Kronecker symbol. Since the operators satisfy the resolution of unity  $\sum_{r=1}^d A_r^\dagger A_r = \mathbb{1}$  the operation is deterministic and one can use the following formulas [3]:

$$F = \frac{1}{d(d+1)} \left( d + \sum_{r=1}^d |\text{Tr} A_r|^2 \right), \quad (7)$$

$$G = \frac{1}{d(d+1)} \left( d + \sum_{r=1}^d \langle a_r | A_r^\dagger A_r | a_r \rangle_E \right). \quad (8)$$

Hence, one obtains using Eq. (6) that

$$F = \frac{1 + (\alpha + \sqrt{d}\beta)^2}{d+1}, \quad G = \frac{1 + (\alpha + \frac{\beta}{\sqrt{d}})^2}{d+1}. \quad (9)$$

Substituting finally these mean fidelities back into the inequality (1) we find that they saturate the inequality. This means that the fidelities (9) lie for any state (4) of ancilla on the very boundary of the quantum mechanically allowed region defined by the inequality (1). Moreover, by changing continuously the parameter  $\alpha$  in this state from 1 to 0 one can continuously move along the whole boundary from its one extreme point  $(G_{\max}, F_{\min}) = (2/(d+1), 2/(d+1))$  to its other extreme point  $(G_{\min}, F_{\max}) = (1/d, 1)$ . Up to now we have considered the device in Fig. 1 without twirling operations  $T$  and  $T^{-1}$ . Such a scheme is not universal as the output state fidelity  $f = \langle \psi | \rho_{\text{out}} | \psi \rangle$  is dependent on the input state  $|\psi\rangle$ . In order to obtain a universal device where the fidelity  $f$  is state independent and therefore  $f = F$  it is sufficient to place the QND interaction in between two twirling operations as is depicted in Fig. 1 [4].

Interestingly, the controllable optimal quantum operation (6) can be implemented using the qudit CNOT gate  $U_{\text{CNOT}}$  defined as  $U_{\text{CNOT}} |i\rangle_S |j\rangle_A = |i\rangle_S |i \oplus j\rangle_A$  where  $\oplus$  denotes addition modulo  $d$  and  $\{|i\rangle_{S,A}\}_{i=1}^d$  are chosen sets of basis states of qudits  $S$  and  $A$  [5]. For the CNOT gate the relevant states of the ancilla satisfying the conditions (2) and (3) are  $|\mu\rangle_A = |0\rangle_A$  and

$|\kappa\rangle_A = (1/\sqrt{d}) \sum_{i=1}^d |i\rangle_A$ , respectively. Eve then measures the ancilla in the basis  $\{|i\rangle_A\}_{i=1}^d$  and prepares the state  $|r\rangle_E$  if she finds the ancilla in the state  $|r\rangle_A$ . Notice, that optimal quantum operation saturating the inequality (1) can be alternatively realized via teleportation scheme with two entangled ancillas [3].

A specific feature of the quantum operation (6) is that it is diagonal in the basis  $\{|a_i\rangle_S\}_{i=1}^d$  and thus it preserves these basis states. Therefore, our scheme can be interpreted as the QND measurement on the qudit  $S$  of some observable (non-demolition variable)  $\mathcal{A} = \sum_{i=1}^d a_i |a_i\rangle_S \langle a_i|$  with non-degenerate eigenvalues  $a_i$ . In fact, the QND measurement preserving the basis states  $\{|a_i\rangle_S\}_{i=1}^d$  can be implemented with a more general class of two-qudit unitary interactions. To illustrate this, suppose Eve is given an unitary interaction  $U$  satisfying only the condition (2) where, in addition, the states of ancilla  $\{|\mu_i\rangle_A\}_{i=1}^d$  are in general nonorthogonal. Clearly, in this case Eve's best strategy is to discriminate among these states and since she has no a priori information about the occurrence of the states (the complex amplitudes  $c_i$  are unknown) the states have equal a priori probabilities. In order to preserve the deterministic character of her operation when in each run of the protocol the measurement on the ancilla uniquely determines which of the basis states  $\{|a_i\rangle_E\}_{i=1}^d$  is to be prepared she has to use an ambiguous discrimination [6] of the states  $\{|\mu_i\rangle_A\}$ . In this approach she applies a generalized measurement  $\Pi_i$ ,  $i = 1, \dots, d$  ( $\Pi_i \geq 0, \sum_{i=1}^d \Pi_i = \mathbb{1}$ ) on the ancilla  $A$  discriminating among these states and prepares the state  $|a_r\rangle_E$  if she detected  $\Pi_r$ . Making use of the Eqs. (7) and (8) one then finds that

$$F = \frac{2 + \frac{1}{d} \sum_{i \neq j=1}^d A \langle \mu_i | \mu_j \rangle_A}{d+1}, \quad G = \frac{2 - P_e}{d+1}, \quad (10)$$

where  $P_e = 1 - (1/d) \sum_{r=1}^d A \langle \mu_r | \Pi_r | \mu_r \rangle_A$  is Eve's error rate. The present quantum operation apparently preserves the basis states  $\{|a_i\rangle_S\}_{i=1}^d$  and therefore it can be again interpreted as a QND measurement. The question that can be risen in this context is whether also this QND measurement allows to gain maximum possible information on the input state similarly as in the previously discussed scheme. The Eq. (10) reveals that this is not the case as soon as the states  $\{|\mu_i\rangle_A\}$  are nonorthogonal. Namely, Eve's error rate is always greater than zero for nonorthogonal states, i. e.  $P_e > 0$ , and therefore Eve's estimation fidelity will be always less than the highest possible value  $G_{\max} = 2/(d+1)$  [7]. Another interesting question is whether and when the fidelities (10) saturate the inequality (1). Since the fidelity  $F$  in Eq. (10) is independent of the measurement on the ancilla the corresponding estimation fidelity  $G$  will be maximized if Eve performs optimal discrimination of the ancillary states  $\{|\mu_i\rangle_A\}_{i=1}^d$  that minimizes the error rate  $P_e$ . For  $d > 2$  the minimal error rate can be found only numerically [8]. However, for qubits ( $d = 2$ ) the minimal error rate can be calculated analytically and it is given by the formula

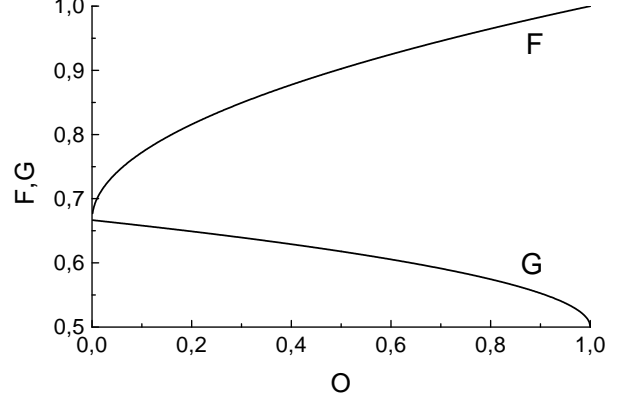


FIG. 2: The dependence of the fidelities  $F$  and  $G$  on the overlap  $O = |\langle \mu_1 | \mu_2 \rangle|^2$ .

$P_{e,\min} = (1 - \sqrt{1 - |\langle \mu_1 | \mu_2 \rangle|^2})/2$  [6]. (In this formula and in what follows we drop the subscript  $A$  in states  $\{|\mu_i\rangle_A\}_{i=1}^d$ ). Hence, using the formula and Eq. (10) one obtains the output fidelity and the estimation fidelity in the form

$$F = \frac{2 + |\langle \mu_1 | \mu_2 \rangle| \cos \phi}{3}, \quad (11)$$

$$G = \frac{3 + \sqrt{1 - |\langle \mu_1 | \mu_2 \rangle|^2}}{6}, \quad (12)$$

where  $\phi = \arg \langle \mu_1 | \mu_2 \rangle$ . Substituting now the fidelities into the inequality (1) for  $d = 2$  one finds that they are saturated only if  $\phi = 2k\pi$ , where  $k$  is an integer. The undesirable phase  $\phi$  can be removed and thus this optimality condition can be established by applying the phase shift  $|a_1\rangle_S \rightarrow |a_1\rangle_S$  and  $|a_2\rangle_S \rightarrow e^{-i\phi} |a_2\rangle_S$  on Bob's qubit after the interaction  $U$ . As a consequence, any two-qubit unitary interaction  $U$  satisfying the condition (2) supplemented by the suitable additional phase shift on a qubit  $S$  can be used for the QND measurement that saturates the inequality (1) for  $d = 2$ . If, however, the states  $\{|\mu_i\rangle\}_{i=1}^2$  are nonorthogonal, it is not possible to achieve the maximum possible estimation fidelity  $G_{\max} = 2/3$  and therefore one cannot move along the whole boundary of the region defined by the inequality (1) for  $d = 2$  by changing the state  $|\mu\rangle_A$  of the ancilla. The dependences of the fidelities  $F$  and  $G$  given in Eqs. (11) and (12) where  $\phi = 0$  on the overlap  $O = |\langle \mu_1 | \mu_2 \rangle|^2$  is depicted in Fig. 2.

However, the maximum possible estimation fidelity can be achieved, at least on a suitable sub-ensemble, if Eve replaces the ambiguous discrimination by the unambiguous discrimination [9] of the states  $\{|\mu_i\rangle\}_{i=1}^2$ . This strategy can be used if the states are linearly independent and it allows to discriminate them perfectly with a certain probability  $P_I$  of inconclusive result. The corresponding generalized measurement has three components  $\Sigma_1 = |\mu_2^\perp\rangle\langle\mu_2^\perp|/(1 + |\langle \mu_1 | \mu_2 \rangle|)$ ,  $\Sigma_2 = |\mu_1^\perp\rangle\langle\mu_1^\perp|/(1 + |\langle \mu_1 | \mu_2 \rangle|)$  ( $|\mu_i^\perp\rangle$  is orthogonal state to the state  $|\mu_i\rangle$ ) and  $\Sigma_0 =$

$\mathbb{1} - \sum_{i=1}^2 \Sigma_i$  ( $\Sigma_i \geq 0$ ). The component  $\Sigma_0$  corresponds to the inconclusive result and this measurement is optimal in the sense that the probability  $P_I$  attains minimum possible value  $P_{I,\min} = |\langle \mu_1 | \mu_2 \rangle|$ . Apparently, if Eve detects the conclusive result  $\Sigma_i$ ,  $i = 1, 2$  then she prepares the state  $|a_i\rangle_E$ . Therefore, on the sub-ensemble corresponding to the conclusive ( $C$ ) results Eve prepares the state  $\rho_{E,C} = \sum_{i=1}^2 |c_i|^2 |a_i\rangle_{EE} \langle a_i|$  for which the mean estimation fidelity achieves maximum possible value, i. e.  $G_C = G_{\max} = 2/3$ . On the same sub-ensemble, Bob receives the same mixed state as prepares Eve whence  $F_C = 2/3$ . The obtained result clearly illustrates that if Eve uses unambiguous discrimination of the states  $\{|\mu_i\rangle\}_{i=1}^2$  than in cases when she detects the conclusive result she is able to obtain the best possible estimate of the state  $|\psi\rangle_S$  even if the QND interaction is imperfect and encodes the information on the state into the nonorthogonal states of ancilla.

To experimentally test this peculiar property of the QND interaction, the experimentalists can use a recent progress in the implementations of the CNOT gates between the atoms and photons. Due to a possible long time for manipulations of qubits (represented by long-lived electronic states) and high efficiency of state detection, trapped and cooled ions are ideally suited for implementations of quantum operations. A single-ion CNOT gate has been realized some time ago in [10]. Recently, a two-ion CNOT gate based on  $^{40}\text{Ca}^+$  ions in a linear Paul trap which were individually addressed using focused laser beams has been implemented [11]. Also two-ion  $\pi$ -phase gate demonstrated with  $^9\text{Be}^+$  ions in a harmonic trap [12] can be used for the same purpose. Further, probabilistic CNOT gates, where the qubits are destroyed upon failure, have been experimentally tested in optical systems. Despite of the fact that these CNOT gates for polarization qubits [13] and path qubits [14] are not deterministic they are sufficient to experimentally prove the fundamental trade-off between the estimation fidelity  $G$  and the output fidelity  $F$ . To achieve universal character of disturbance introduced into the measured state mutually inverse twirling operations [15] have to be implemented on qubit  $S$  before and after the QND interaction (see Fig. 1). In summary, in all the mentioned experimental implementations of the CNOT gates the fidelity trade-off can be thus directly experimentally measured.

There are few important and interesting consequences that have to be noticed. They can stimulate broad discussion and future work. First, our result shows that it is allowed to achieve any optimal fidelity measurement with a minimal disturbance by "programming" the QND interaction by a single program ancillary state. This is an interesting result in the context of a previous proof that it is not possible to programme any single-qubit unitary operation and measurement using only a single qubit program [16, 17]. More generally, any optimal fidelity measurement of a qudit can be programmed by an ancilla with the  $d$ -dimensional Hilbert space. A network for this

optimal fidelity measurement can be proposed, for example, for qudit with  $d = 4$  as it was suggested in our previous work [18]. Second, the effect of ambiguous discrimination of ancillary states outgoing QND interaction has been discussed. As we know there is still no bound on maximal success rate for this kind of measurement on a quantum system. It is an open question if such the optimal measurement can be also based on the QND interaction. Third, in fact we decomposed optimal fidelity measurement with minimal disturbance for a single copy into two steps: programmable QND coupling and discrimination of ancillary states. Thus QND interaction is not only optimal for accessing information encoded in single preferred basis but also it is optimal for universal measurement without a preferred basis. The difference is only in the twirling operation which effectively changes the preferred basis. For many identical copies of input state it is an open question if optimal fidelity measurement is based on the same method. Can be this strategy used to approach optimal fidelity measurement on many identical copies [19]? At the end, the problem of the quantum complementarity and erasure for the QND coupling is closely related [20, 21]. It is known that perfect two-qubit QND coupling with arbitrary pure ancillary state is perfectly reversible if Eve implements an appropriate measurement and Bob performs according to the measurement results an appropriate unitary operation. The remaining problem to be discussed is also how the imperfect QND coupling can be reversed using only local operations and classical communication.

In this article, a fundamental property of the QND measurement came to light: performing QND measurement randomly along all basis in Hilbert space of the system the tightest bound (1) between the estimation fidelity  $G$  and the output fidelity  $F$  of the measured system can be saturated. To change optimal ratio between these fidelities it is sufficient to control the ancilla of the QND measurement. Even when the used QND measurement is not perfect we can still optimally control the fidelity trade-off but only in a restricted range if the output ancillary states are optimally ambiguously discriminated. These results are not only important from the fundamental point of view but they can be used to distribute information carried by quantum state without any preferred basis.

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